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THE NORTH SEA PROBLEM II

Influence of a stationary windfield upon a bay with a uniform depth

by

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1. Introduction

This paper is the second of a series of papers under the heading "The North Sea Problem". In this paper we shall study the equilibrium situation which arises when a rectangular bay with a uniform depth is subjected to a stationary windfield. We shall use the same notation as in the previous paper. References to the latter paper will be indicated by I followed by the section number or formula number. At the end of this paper a short list of symbols and a concise bibliography is given. Further literature is given in I.

The mathematical model (cf. I 1) of the North Sea is a rectangle which in Cartesian coordinates x,y is given by $0 < x < a$ and $0 < y < b$. The sides $x=0$, $x=a$ and $y=0$ represent coasts, the side $y=b$ represents the open end at the ocean. The side $y=0$ corresponds roughly to the Dutch coast and the middle $(\frac{1}{2}a,0)$ roughly to the position of Hoek van Holland. For simplicity the direction of the positive Y-axis will be called the northern direction. However, the longitudinal axis of the North Sea makes the slight deviation of about 24° with the geographic North. In this paper we have restricted ourselves mainly to the determination of the elevation at the "Dutch" coast $y=0$ and in particular to that of the middel $(\frac{1}{2}a,0)$. The numerical data of the model are chosen according to the physical situation of the North Sea. They are given in the following section.

We shall consider in particular the influence of a "linear" windfield of the type (5.1) to the elevation at the "Dutch" coast. The reader is reminded of the fact that for this type of windfield not the velocity of the wind but rather the surface stress is a linear function of the place. The divergence and the rotation of the surface stress are constants. Some results are also valid for non-linear windfields for which the rotation is a constant.

The elevation at the "Dutch" coast due to the linear windfield (5.1) is given by table (6.1) and formula (6.7). In order to appreciate the influence of the rotation of the Earth Ω and the bottom friction λ we have calculated the elevation at "Hoek van Holland" $(\frac{1}{2}a,0)$ which would occur if either $\Omega/\lambda = 0$ or $\Omega/\lambda = \infty$. The results are given by the formulae (6.8) and (6.9). The following conclusions can be drawn

1. For a uniform N-S wind the elevation at the "Dutch" coast does not depend on the rotation of the earth (Ω).
2. For a uniform W-E wind the elevation at the "Dutch" coast is rather influenced by Ω.
3. For the given model the most unfavourable direction of a uniform wind as regards the elevation at the "Dutch" coast is about 12° NNW.
4. The influence of Ω upon the contributions of the divergence terms U_1 and V_2 to the elevation at the "Dutch" coast is rather small.
5. The influence of Ω upon the contributions of the rotation terms U_2 and V_1 to the elevation at the "Dutch" coast is very large.

These conclusions must be considered with some reservation since they relate to the mathematical model rather than to the North Sea proper. It is expected that, at least for some windfields, the sloping bottom of the North Sea (cf. I 1) has a strong influence upon the elevation of the sea surface. In the following paper a model will be discussed for which the depth is an exponential function of y only. The latter model which is a much better representation of the North Sea basin permits a mathematical treatment which is very similar to that of the present model.

The problem of the equilibrium state has been considered by a number of writers. Schalkwijk (1) considers a rectangular bay with a uniform depth bordering on an ocean with the same depth. His results with respect to the "Dutch" coast are confirmed by Veltkamp (1) who gave an exact treatment by using functiontheoretical methods. The latter author also considers (2) the case of a rectangular bay with a uniform depth bordering on an ocean with a much greater depth. For a linear windfield he gives explicit results which are almost identical with ours. We also mention a study of Weenink (1) who considers Schalkwijk's model with a uniform northern windfield which is absent in the eastern half of the bay. A summary of Veltkamp's work is also given in Van Dantzig's congress paper (1).

2. The mathematical problem

The stationary state of the rectangular bay $0 < x < a$, $0 < y < b$ is governed by the equations

$$(2.1)\qquad \begin{cases} \lambda u - \Omega v + c^2 \dfrac{\partial \zeta}{\partial x} = U \\ \lambda v + \Omega u + c^2 \dfrac{\partial \zeta}{\partial y} = V \end{cases}$$

and

$$(2.2)\qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

with the boundary conditions

$$(2.3)\qquad \begin{cases} u=0 & \text{at } x=0 \text{ and } x=a \\ v=0 & \text{at } y=0 \\ \zeta=0 & \text{at } y=b. \end{cases}$$

These equations can be obtained from(I 2.6) and (I 2.7) by omitting the terms with the time derivative.
In this model of the North Sea we have the following approximate numerical data

$a = 400$ km	$\Omega = 0.44\ h^{-1}$
$b = 800$ km	$\lambda = 0.09\ h^{-1}$
$h = 65$ m	$c = 91$ km/h.

The relation between the absolute value of the frictional force of the wind (U,V) and the velocity of the wind at sealevel is according to (I 2.3) given by

$$(2.4)\qquad \sqrt{U^2+V^2} = 3.0 \times 10^{-6}\ v_s^2 .$$

Some simplification is obtained if x and y are measured in units of a/π . This is equivalent to putting formally $a=\pi$. The calculations will be performed with $\lambda = \Omega/5$.

3. Method of solution

The rotation of the windfield R is defined by

(3.1) $$R \overset{\text{def}}{=} \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} .$$

If R=0 and if U(x,b)=0 it can be shown quite easily that in the stationary state there is no stream. For in that case the solution of (2.1), (2.2) and (2.3) is simply given by

(3.2) $$u=v=0, \qquad c^2 \zeta(x,y) = -\int_y^b V(x,\eta)\,d\eta .$$

Otherwise, however, there exists a stream and we may introduce a non-vanishing streamfunction $\phi(x,y)$ by means of

(3.3) $$\lambda u = -\frac{\partial \phi}{\partial y} , \qquad \lambda v = \frac{\partial \phi}{\partial x} .$$

At the coastline x=0, x=π and y=0 we may take $\phi=0$ since the coast is a streamline.

From (2.1) we obtain by elimination of ζ and by using (3.3)

(3.4) $$\Delta \phi = R.$$

To this partial differential equation we may add

a the coast condition

(3.5) $$\phi = 0 \qquad \text{at } x=0,\ x=\pi,\ y=0;$$

b the ocean condition

(3.6) $$\frac{\partial \phi}{\partial y} + \mathrm{tg}\,\mu\pi\, \frac{\partial \phi}{\partial x} = -U \quad \text{at} \quad y=b,$$

with

(3.7) $$\mu \overset{\text{def}}{=} \frac{1}{\pi} \,\mathrm{arctg}\, \frac{\Omega}{\lambda} .$$

The solution of this problem will be written in the form

(3.8) $$\phi(x,y) = \phi_0(x,y) + \phi_1(x,y),$$

where ϕ_0 is a particular solution of (3.4) satisfying the boundary conditions at x=0 and x=π, but not necessarily those at y=0 and y=b.

Therefore ϕ_1 satisfies the Laplace equation

(3.9) $$\Delta \phi_1 = 0,$$

and the boundary conditions

(3.10) $$\phi_1 = 0 \quad \text{at} \quad x=0 \text{ and } x=\pi .$$

The elementary solutions of (3.9) and (3.10) are of the form

(3.11) $$\sin nx\ e^{\pm ny},$$

for n=1,2,3,...
Hence we may put

(3.12) $$\phi_1(x,y) = \sum_{n=1}^{\infty} c_n \sin nx\ e^{-ny} - \sum_{n=1}^{\infty} d_n \sin nx\ e^{-n(b-y)}.$$

A particular solution ϕ_o of (3.4) can be found without much difficulty. In this paper we shall consider only the case of a constant rotation. If R is a constant a suitable particular solution is given by

(3.13) $$\phi_o = -\tfrac{1}{2}Rx(\pi-x).$$

If, however, R is a given function of x and y a particular solution ϕ_o can be found by solving the following problem of Green

(3.14) $$\Delta G(x,y,x_o,y_o) = -\delta(x-x_o)\,\delta(y-y_o),$$

(3.15) $$G = 0 \quad \text{for } x=0 \text{ and } x=\pi.$$

This can be solved in various ways. However, the discussion of this problem will be omitted here.

We shall next direct our attention to the conditions at y=0 and y=b which eventually will lead to the determination of the coefficients c_n and d_n. From (3.8), (3.12) and (3.13) it follows that

(3.16) $$\phi(x,y) = -\tfrac{1}{2}Rx(\pi-x) + \sum_{n=1}^{\infty} c_n \sin nx\ e^{-ny} - \sum_{n=1}^{\infty} d_n \sin nx\ e^{-n(b-y)}.$$

In the following discussion it will always be assumed that b is so large that the boundary conditions at y=0 and y=b do not interact. The coefficients c_n and d_n are roughly of the same order. However, the exponential factors exp-nb are very small. In the numerical application with the North Sea data we have the dimensionless value $b=2\pi$. We note that $\exp -2\pi = 0.002$, so that this value of b can be considered indeed as large.

The coast condition at y=0 gives

(3.17) $$\sum_{n=1}^{\infty} c_n \sin nx = \tfrac{1}{2}Rx(\pi-x) + \sum_{n=1}^{\infty} e^{-nb} d_n \sin nx.$$

The contribution of the series on the right-hand side can be neglected

in the numerical application with $b=2\pi$.
We shall write symbolically

$$(3.18)\qquad \sum_{n=1}^{\infty} c_n \sin nx = \tfrac{1}{2}Rx(\pi-x) + O(e^{-b}).$$

Then, with the neglection of the order term, we obtain from (3.18)

$$(3.19)\qquad \begin{cases} c_n = 0 & \text{for even } n, \\ c_n = \dfrac{4R}{\pi n^3} & \text{for odd } n. \end{cases}$$

The ocean condition at $y=b$ gives with a similar order term

$$(3.20)\qquad \sum_{n=1}^{\infty} n\, d_n \sin(nx+\mu\pi) = \cos\mu\pi\; U(x,b) - \sin\mu\pi\; R(\tfrac{1}{2}\pi - x) + O(e^{-b}).$$

Expansions of this kind have been studied in Lauwerier (2). A survey of the properties of this special kind of trigonometrical expansion will be given in the following section.
Here only the following asymptotic expression will be mentioned

$$(3.21)\qquad d_n = \frac{(-1)^{n-1}}{n^{2-2\mu}} D' + \frac{1}{n^{2+2\mu}} D'' + O(n^{-4+2\mu}).$$

We shall consider in particular the elevation of the sea at the coast $y=0$ in view of its importance for the North Sea problem. From the first equation of (2.1) it follows for $y=0$

$$(3.22)\qquad c^2 \frac{\partial \zeta}{\partial x} = U(x,0) + \frac{\partial \phi}{\partial y},$$

so that by integration an expression for the relative elevation at $y=0$ can be obtained

$$(3.23)\qquad c^2 \{\zeta(x,0) - \zeta(0,0)\} = \int_0^x U(\xi,0)\,d\xi + \int_0^x \frac{\partial}{\partial y}\phi(\xi,0)\,d\xi .$$

Substitution of (3.16) with (3.18) gives

$$(3.24)\qquad c^2 \{\zeta(x,0) - \zeta(0,0)\} = \int_0^x U(\xi,0)\,d\xi - \frac{4R}{\pi} \sum{}_1 \frac{1-\cos nx}{n^3} + O(e^{-b}),$$

where $\sum_1$ denotes a summation over odd indices. From the second equation of (2.1) it follows for $x=0$ that

$$(3.25)\qquad c^2 \frac{\partial \zeta}{\partial y} = V(0,y) - \frac{\partial \phi}{\partial x},$$

so that by integration an expression for the absolute elevation at

(0,0) can be obtained

(3.26) $$c^2\varphi(0,0) = -\int_0^b V(0,\eta)d\eta - \tfrac{1}{2}\pi bR + \frac{4R}{\pi}\sum_1 \frac{1}{n^3} - \sum_{n=1}^{\infty} d_n + O(e^{-b}).$$

From (3.24) and (3.26) it follows that

(3.27) $$c^2\varphi(x,0) = \int_0^x U(\xi,0)d\xi - \int_0^b V(0,\eta)d\eta +$$
$$+ R\{c(x) - \tfrac{1}{2}\pi b\} - \sum_{n=1}^{\infty} d_n + O(e^{-b}),$$

where

(3.28) $$c(x) \overset{\text{def}}{=} \frac{4}{\pi}\sum_1 \frac{\cos nx}{n^3} .$$

A few values of the function c(x) are given in the following table

$x = m\pi/12$	$c(x)$
0	1.339
1	1.262
2	1.091
3	0.863
4	0.595
5	0.303
6	0

table 3.1

4. The ocean condition

The ocean condition (3.20) is of the form

$$(4.1) \qquad f(x) = \sum_{n=1}^{\infty} b_n \sin(nx+\mu\pi), \qquad 0 < x < \pi ,$$

where $f(x)$ is a given function. We shall assume that μ is real and that in particular $0 < \mu < \frac{1}{2}$.

An important special case of (4.1) can be obtained in the following way. We start from the power series

$$(4.2) \qquad \left(\frac{1+s}{1-s}\right)^{\beta} = \sum_{n=0}^{\infty} e_n(\beta)\, s^n , \qquad |s| < 1$$

where β is an arbitrary constant. The first few coefficients are

$$(4.3) \qquad e_o = 1 \qquad e_1 = 2\beta \qquad e_2 = 2\beta^2 \quad \ldots$$

and we note that

$$(4.4) \qquad e_n(-\beta) = (-1)^n\, e_n(\beta).$$

If in (4.2) the substitution $s=\exp ix$ and $\beta=-2\mu$ is made we find by separating the real and the imaginary part

$$(4.5) \qquad \cos\mu\pi\, (\mathrm{tg}\tfrac{1}{2}x)^{2\mu} = \sum_{n=0}^{\infty} e_n(-2\mu)\cos nx,$$

and

$$(4.6) \qquad \sin\mu\pi\, (\mathrm{tg}\tfrac{1}{2}x)^{2\mu} = -\sum_{n=1}^{\infty} e_n(-2\mu)\sin nx.$$

From these two expansions it follows that

$$(4.7) \qquad \sum_{n=0}^{\infty} e_n(-2\mu)\sin(nx+\mu\pi) = 0,$$

or

$$(4.8) \qquad \sin\mu\pi = -\sum_{n=1}^{\infty} e_n(-2\mu)\sin(nx+\mu\pi),$$

which covers the case of $f(x) \equiv 1$.

From (4.5) and (4.6) it follows that

$$(4.9) \quad \tfrac{1}{2}\pi\, e_n(-2\mu) = \cos\mu\pi \int_0^{\pi} (\mathrm{tg}\tfrac{1}{2}t)^{2\mu} \cos nt\, dt =$$

$$= -\sin\mu\pi \int_0^{\pi} (\mathrm{tg}\tfrac{1}{2}t)^{2\mu} \sin nt\, dt.$$

so that

$$(4.10) \qquad \int_0^{\pi} (tg\tfrac{1}{2}t)^{2\mu} \cos(nt-\mu\pi)dt = 0$$

for n=1,2,3...

If now μ is changed into $\frac{1}{2}-\mu$ we obtain the important result

$$(4.11) \qquad \int_0^{\pi} (tg\tfrac{1}{2}t)^{1-2\mu} \sin(nt+\mu\pi)dt = 0$$

for n=1,2,3...

This means that the set $\sin(nx+\mu\pi)$ n=1,2,3,... is not complete - in a certain sense - but that all elements of the set are orthogonal to the function $(tg\frac{1}{2}x)^{1-2\mu}$. Now the following results will be quoted from Lauwerier (2).

For $0 \leq \mu < \frac{1}{2}$ the expansion (4.1) is always possible and unique. The asymptotic behaviour of the coefficients is of the subharmonic order $n^{-1+2\mu}$. In general the convergence is not uniform in (0,1). However, if

$$(4.12) \qquad \int_0^{\pi} (tg\tfrac{1}{2}t)^{1-2\mu} f(t)dt = 0,$$

the asymptotic behaviour of the coefficients is of the hyper-harmonic order $n^{-1-2\mu}$. Then the convergence is absolute and uniform in (0,1).

For $0 < \beta < 1$ the asymptotic behaviour of the coefficients $e_n(\beta)$ is given by

$$(4.13) \qquad e_n(\beta) = \frac{2^{\beta}}{\Gamma(\beta)n^{1-\beta}} + \frac{(-1)^{n-1}2^{-\beta}}{\Gamma(-\beta)n^{1+\beta}} + O(n^{-3+\beta}).$$

For the coefficients b_n of (4.1) the following asymptotic expression can be derived

$$(4.14) \qquad b_n = -\frac{e_n(-2\mu)}{\pi} \int_0^{\pi} (tg\tfrac{1}{2}t)^{1-2\mu} f(t)dt + \frac{B'}{n^{1+2\mu}} + O(n^{-3+2\mu}),$$

where B' is a constant.

In this paper the coefficients of the expansion of a linear function only need be known. Let the coefficient g_n be determined by the expansion

$$(4.15) \qquad \sum_{n=1}^{\infty} g_n \sin(nx+\mu\pi) = \sin\mu\pi \, (\tfrac{1}{2}\pi - x).$$

The non-uniformly convergent series on the left-hand side of (4.15) can be converted into an absolutely uniformly convergent series by writing

(4.16) $$\sum_{n=1}^{\infty}\left\{g_n+g_o\, e_n(-2\mu)\right\} \sin(nx+\mu\pi) = \sin\mu\pi\,(\tfrac{1}{2}\pi-x-g_o),$$

where

(4.17) $$g_o \overset{\text{def}}{=} \frac{\sin\mu\pi}{\pi}\int_0^{\pi}(\mathrm{tg}\tfrac{1}{2}t)^{1-2\mu}\,(\tfrac{1}{2}\pi-t)\,dt.$$

The new series can be differentiated term by term. This gives

(4.18) $$\sum_{n=1}^{\infty} n\left\{g_n + g_o\, e_n(-2\mu)\right\} \cos(nx+\mu\pi) = -\sin\mu\pi\ .$$

If this is compared with (4.8) it follows at once that

(4.19) $$g_n = -g_o e_n(-2\mu) + \mathrm{tg}\,\mu\pi\, \frac{e_n(1-2\mu)}{n}\ .$$

Hence the coefficients g_n can be calculated numerically with the single quadrature (4.17).

5. A linear windfield

We shall consider in particular a windfield the surface stresses of which are a linear function of x and y. We shall take

$$(5.1)\qquad \begin{cases} U = U_o + U_1(1-\frac{2x}{\pi}) + U_2(1-\frac{y}{b}) \\ V = V_o + V_1(1-\frac{2x}{\pi}) + V_2(1-\frac{y}{b}) \;. \end{cases}$$

This windfield has the constant rotation

$$(5.2)\qquad R = \frac{1}{b}U_2 - \frac{2}{\pi}V_1 \;.$$

The influence of the terms V_o and V_2 follows already from (3.2). The expression for the absolute elevation (3.27) becomes

$$(5.3)\qquad c^2\zeta(x,0) = xU_o+(x-x^2/\pi)U_1+(x-\tfrac{1}{2}\pi)U_2-bV_o-\tfrac{1}{2}bV_2 + \\ + R\,c(x) - \sum_{n=1}^{\infty} d_n + O(e^{-b}),$$

where the coefficients d_n are determined by the ocean condition (3.20) which here explicitly reads

$$(5.4)\qquad \sum_{n=1}^{\infty} n\, d_n \sin(nx+\mu\pi) = \cos\mu\pi\; U_o+(\tfrac{2}{\pi}U_1\operatorname{cotg}\mu\pi - R)\sin\mu\pi(\tfrac{1}{2}\pi-x)+ \\ +O(e^{-b}).$$

It follows from (4.8) and (4.15) that

$$(5.5)\qquad d_n = -\frac{\lambda\, e_n(-2\mu)}{\Omega n}U_o + \left(\frac{2\lambda}{\pi\Omega}U_1-R\right)\frac{g_n}{n} + O(e^{-b}).$$

In view of (5.3) we need only the sum of the series $\sum d_n$. This involves the computation of

$$(5.6)\qquad \sum_{n=1}^{\infty} n^{-1}e_n(-2\mu) \qquad \text{and} \qquad \sum_{n=1}^{\infty} n^{-1}g_n \;.$$

For the first sum a simple analytic expression can be derived

$$(5.7)\qquad \sum_{n=1}^{\infty} n^{-1}\, e_n(-2\mu) = \psi(\tfrac{1}{2}) - \psi(\tfrac{1}{2}+\mu),$$

where

$$(5.8)\qquad \psi(x) \overset{\text{def}}{=} \Gamma'(x)/\Gamma(x).$$

For the second sum we may write in view of (4.19)

$$(5.9)\qquad \sum_{n=1}^{\infty} n^{-1}g_n = -g_o\sum_{n=1}^{\infty} n^{-1}e_n(-2\mu)+\operatorname{tg}\mu\pi\sum_{n=1}^{\infty} n^{-2}e_n(1-2\mu).$$

The second sum on the right-hand side must be summed numerically.

6. Numerical calculations

We shall write

(6.1) $$E = -\operatorname{cotg}\mu\pi \sum_{n=1}^{\infty} n^{-1} e_n(-2\mu),$$

and

(6.2) $$G = \operatorname{cotg}\mu\pi \sum_{n=1}^{\infty} n^{-1} g_n.$$

For $\Omega/\lambda = 5$ it follows from (4.17) by numerical integration and from (5.7) and (5.9) that

(6.3) $$\begin{cases} g_o = -0.169 \\ E = 0.256 \\ G = 0.232 \, . \end{cases}$$

From (5.3) and (5.5) we obtain for the absolute elevation at $y=0$ explicitly

(6.4) $$\begin{aligned} c^2 \zeta(x,0) = {} & (x-E)U_o + \frac{2}{\pi}\left\{\tfrac{1}{2}x(\pi-x) - G\right\} U_1 + \\ & + \frac{1}{b}\left\{c(x) + G \operatorname{tg}\mu\pi + b(x-\tfrac{1}{2}\pi)\right\} U_2 + \\ & - b\, V_o - \frac{2}{\pi}\left\{c(x)+G \operatorname{tg}\mu\pi\right\} V_1 - \tfrac{1}{2}b\, V_2. \end{aligned}$$

It may be of interest to give also the corresponding expressions in the extreme cases $\Omega/\lambda = 0$ and $\Omega/\lambda = \infty$.

a $\Omega/\lambda = 0$.

This means that $\mu \to 0$. It can easily be derived that then

$$E \to \tfrac{1}{2}\pi \quad \text{and} \quad G \to \frac{1}{12}\pi^2.$$

Hence (6.4) gives

(6.5) $$\begin{aligned} c^2 \zeta(x,0) = {} & (x-\tfrac{1}{2}\pi)U_o + \frac{1}{\pi}\left\{(x-\tfrac{1}{2}\pi)^2 + \frac{1}{12}\pi^2\right\} U_1 + \\ & + \frac{1}{b}\left\{c(x)+b(x-\tfrac{1}{2}\pi)\right\} U_2 + \\ & - b\, V_o - \frac{2}{\pi} c(x)\, V_1 - \tfrac{1}{2}b\, V_2. \end{aligned}$$

b $\Omega/\lambda = \infty$.

This means that $\mu \to \frac{1}{2}$. In this case we have

$$E \to 0 \quad \text{and} \quad G \operatorname{tg}\mu\pi \to c(0).$$

Hence (6.4) gives

$$c^2 \varphi(x,0) = x\,U_0 + \frac{1}{\pi}\,x(\pi-x)U_1 + \frac{1}{b}\left\{c(x)-c(\pi)+b(x-\tfrac{1}{2}\pi)\right\}U_2 + \tag{6.6}$$
$$- b\,V_0 - \frac{2}{\pi}\left\{c(x)-c(\pi)\right\}V_1 - \tfrac{1}{2}b\,V_2 .$$

In the following table the values of $c^2\varphi(x,0)$ for $x=0\ (\frac{1}{8}\pi)\pi$ are given according to (6.4) for each component U_j, V_j $(j=1,2,3)$.

$8x/\pi$	U_0	U_1	U_2	V_0	V_1	V_2
0	-0.26	-0.15	-1.17	-2π	-1.59	$-\pi$
1	0.13	0.19	-0.80	-2π	-1.49	$-\pi$
2	0.52	0.44	-0.46	-2π	-1.29	$-\pi$
3	0.92	0.59	-0.13	-2π	-1.03	$-\pi$
4	1.31	0.63	0.19	-2π	-0.74	$-\pi$
5	1.70	0.59	0.51	-2π	-0.45	$-\pi$
6	2.10	0.44	0.84	-2π	-0.19	$-\pi$
7	2.49	0.19	1.18	-2π	0.02	$-\pi$
8	2.88	-0.15	1.56	-2π	0.05	$-\pi$

table 6.1

In order to appreciate the influence of the parameter Ω/λ the value of $c^2\varphi(x,0)$ at $x=\frac{1}{2}\pi$ will be given in the three cases $\Omega/\lambda = 5,\ 0,\ \infty$.

(6.7) $\Omega/\lambda = 5$

$$c^2\varphi(\tfrac{1}{2}\pi,0) = 1.31U_0 + 0.63U_1 + 0.19U_2 + -6.28V_0 - 0.74V_1 - 3.14V_2 .$$

(6.8) $\Omega/\lambda = 0$

$$c^2\varphi(\tfrac{1}{2}\pi,0) = 0.26U_1 + -6.28V_0 - 3.14V_2$$

(6.9) $\Omega/\lambda = \infty$

$$c^2\varphi(\tfrac{1}{2}\pi,0) = 1.57U_0 + 0.78U_1 + 0.21U_2 + -6.28V_0 - 0.85V_1 - 3.14V_2 .$$

These results demonstrate the importance of the Coriolis effect. Since the results (6.7) and (6.9) do not differ very much it can be expected that the elevation at y=0 is not very sensitive to small variations of the value of the coefficient of friction. Veltkamp (2) considers a slightly different model. He assumes that the ocean has a finite depth which is large with respect to that of the bay, and he takes the slight larger value $\Omega/\lambda = 5.24$. His result, when translated in our notation, is

$$(6.10) \qquad c^2 \zeta(\tfrac{1}{2}\pi,0) = 1.32U_0 + 0.64U_1 + 0.17U_2 + \\ -6.28V_0 - 0.74V_1 - 3.14V_2 ,$$

which is in extremely good agreement with our result (6.7).

The results obtained above can be used to calculate ζ in meters for a given field of wind-velocities. It will be sufficient to consider the expression (6.7) only. If again x is measured in kilometers the left-hand side of (6.7) must be replaced by $\pi a^{-1}c^2\zeta(\tfrac{1}{2}a,0)$. The numerical value of the factor $\pi a^{-1}c^2$ is 5.0×10^{-3} m/sec^2. The relation between the absolute value of the windstress and the velocity of the wind at sealevel is given by (2.4). If by way of illustration

$$(6.11) \qquad U = 0 \quad \text{and} \quad V = V_0,$$

and if there is a northern wind of 30m/sec then it follows from (2.4) that $V_0 = -2.7 \times 10^{-4}$.
The expression (6.7) gives

$$(6.12) \qquad 5.0 \times 10^{-3}\, \zeta(\tfrac{1}{2}a,0) = -6.28V_0,$$

so that finally

$$(6.13) \qquad \zeta(\tfrac{1}{2}a,0) = 3.40 \text{ m}.$$

As another application of (6.7) we shall determine the most unfavourable direction for a uniform wind.
Let us take

$$(6.14) \qquad \begin{cases} U = U_0 = w \sin \delta \\ V = V_0 = -w \cos \delta , \end{cases}$$

where w measures the absolute velocity of the wind which is constant and where δ represents the deviation with respect to the longitudinal axis. Formula (6.7) gives

$$(6.15) \qquad c^2 \zeta(\tfrac{1}{2}\pi,0) = 1.31 \sin \delta + 6.28 \cos \delta .$$

The right-hand side has the maximum 6.38 for $\delta = 12^\circ$ approximately. For the North Sea this would mean that a Northwestern wind of about 36° to the North is the most unfavourable one. Further applications of the above-given results are left to the reader.

List of symbols

x, y	Cartesian coordinates. The bay is determined by $0 < x < a$, $0 < y < b$;
u, v	the components of the total stream;
ζ	the elevation of the water surface;
U, V	the components of the surface stress due to the windfield;
U_0, U_1, U_2	the coefficients of a linear windfield for the U component;
V_0, V_1, V_2	the coefficients of a linear windfield for the V components;
Ω	the coefficient of Coriolis;
λ	a coefficient of friction;
h	the depth;
g	the constant of gravity;
c	$c = \sqrt{gh}$;
v_s	the velocity of the wind at sealevel;
$\phi(x, y)$	the streamfunction;
R	the rotation of the windfield;
μ	$\mu = \pi^{-1} \operatorname{arctg} (\Omega / \lambda)$.

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